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A note on caustics and two-dimensional hot spots in microwave heating

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Abstract. Very recently the problem of microwave heating by millimetric waves has been considered numerically and experimentally. It was found that hot spots are produced around the caustic regions of the wave field. In this note two-dimensional hot-spot solutions to the microwave heating problem are constructed in this range of scales by a combination of geometric optics and layer solutions to nonlinear heat equations. It is shown that hot spots are localized along caustics and simple analytic expressions for their shape are derived. Since the problem solved is local, the scalings obtained apply to general situations that involve caustics and heat layers. This simple analysis explains the shape and positions of high-temperature regions observed in experiments and full numerical calculations.

Key words: caustics, geometric, hotspots, microwaves

1. Introduction

The problem of homogeneous heating of materials lead to the study of heating using millimetric waves. The premise is that short-wave electromagnetic radiation will provide homogeneous heating. However, waves in this range do form caustics as a result of the interaction between the wave and the shape of the container. In [1], [2] an approach combining ray tracing with numerical solutions of the heat equation is used to study the effect of caustic formation on the heating pattern. It was found that the temperature patterns are controlled by the geometry of the caustics. However, no mathematical analysis was given. The patterns found in [1] and [2] are very different from the patterns studied in [3]. In [3] the patterns are produced in a relatively smooth electric field via thermal instabilities while in [1] and [2] they are generated by the caustics of the electric field.

It is the purpose of this note to consider the shape of the hot spot produced by a caustic of the incident radiation.

The experimental evidence and numerical simulations of Feher [1], [2], show that the hot spots are wedge-shaped and localized in a boundary-layer fashion around the caustics of the wave field. This suggests a mechanism for high heating at the caustic with the temperature diffusing around it.

We will show, in a drastically simplified model, that hot spots do form around caustics and calculate their shape in terms of the local field at the caustic and the caustic geometry. This calculation shows the basic scalings to be expected in generic situations. This construction clarifies the simple mechanism for hot-spot formation. This note is organized as follows. The second section formulates a simplified model of the physics in the same spirit as in [4]. In

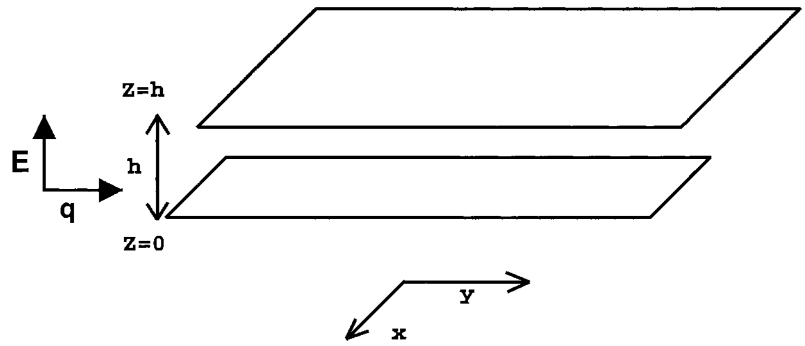

Figure 1. The slab geometry of the heated material.

the third section the hot spots are calculated and a qualitative comparison with experiments is given. The last section is devoted to conclusions.

2. Formulation and simplifications

Since we are interested in fields at caustics we consider the geometry shown in Figure 1. For simplicity we have two parallel plates of infinite extent separated by a small (relative to the size of the plates) distance h. The vertical variable is labelled by z and the horizontal variables by x and y. The plane at $z = 0$ of the sample is assumed to the insulated, *i.e.* no heat flows. The upper part is assumed to radiate. The material is assumed to have a temperature-dependent conductivity.

In this geometry we assume an electromagnetic wave heating the sample. Then the electric field satisfies Maxwell's equations. In the limit of small conductivity and weak temperature dependence of the parameters ϵ and μ the electric and magnetic fields decouple. We thus have from [4] and [5]

$$\begin{aligned} &\mathbf{E}_{tt} \; -c_0^2 \Delta \mathbf{E} + \frac{1}{\epsilon}\sigma(T)\mathbf{E}_t = \mathbf{0}, \\ &\mathbf{H}_{tt} \; -c_0^2 \Delta \mathbf{H} = \mathbf{0}, \end{aligned} \tag{1}$$

where $\mathbf{E}$ is the electric field and $\mathbf{H}$ the magnetic field. The temperature is denoted by T.

Since the cavity is of infinite extent, there are no boundary conditions in the x, y variables. On the faces $z = 0$ and $z = h$ we have $\mathbf{E} \times \mathbf{n} = \mathbf{0}$, where $\mathbf{n}$ is the normal to the surface. The equation for the temperature field T given in [4] is

$$\frac{\partial T}{\partial t} = \nu \Delta T + \gamma(T) P, \tag{2}$$

where P is the average of $|E|^2$ over a microwave period. The initial conditions are prescribed and the boundary condition at $z = 0$ is $\partial T/\partial z = 0$, while the losses at $z = h$ are accounted for by $\nu \partial T/\partial z = -L(T)$ at $z = h$. The parameter ν is the thermal diffusivity, γ is the temperature-dependent rate at which the material absorbs the microwave heat. The nonlinear function $L(T)$ represents the losses.

To simplify Equation (2) we follow Kriegsmann ([3], [6]) using the average equations. Integration of Equation (2) in the z-variable and approximation of the temperature by its value at the midpoint $z = \frac{h}{2}$, gives, denoting again by $T(x, y, t)$ the temperature $T(x, y, h/2, t)$,

$$\frac{\partial T}{\partial t} = \nu\left(\frac{\partial T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \frac{L(T)}{h} + \frac{\gamma(T)}{h}\int_0^h |E|^2 \mathrm{d}z, \tag{3}$$

which is now a two-dimensional equation.

The electric field is chosen to satisfy the boundary conditions at the top and bottom walls. A simple choice is $\mathbf{E} = (0, 0, E(x, y))$, where the boundary conditions are satisfied and the scalar E satisfies the two-dimensional wave equation

$$E_{tt} - c_0^2 \Delta E + \frac{1}{\epsilon}\sigma(T)E_t = 0. \tag{4}$$

Equations (3) and (4) provide the simplified two-dimensional system we are interested in studying. One further simplification will be made in the heat Equation (3).

Since we are interested in solutions for E that are independent of z, the nonlinear source in (3) takes the form:

$$\frac{L(T)}{h} - \gamma(T)|E|^2. \tag{5}$$

It was shown in [4], and used thereafter, that (5) has a bistable behavior as a function of T. In [4] this behavior was conveniently described by a cubic in the form,

$$\Lambda(T_0 - T)(T^2 - T_0 T + \tilde{\gamma}|E|^2). \tag{6}$$

We now take the same choice. Then the model to be studied is:

$$E_{tt} - c_0^2 \Delta E + \frac{1}{\epsilon}\sigma(T)E_t = 0, \tag{7}$$

$$T_t = \nu(T_{xx} + T_{yy}) + \Lambda(T_0 - T)(T^2 - T_0 T + \tilde{\gamma}|E|). \tag{8}$$

The conductivity is taken to be linear in the temperature, $\sigma(T)/\epsilon = \sigma_0 + \sigma_1 T$.

3. Caustic and hot-spot solutions

We will be interested in constructing steady solutions for the heat equation sustained by forced oscillations of the electric field. To this end we proceed as in [4] using nondimensional variables $\tilde{t} = wt, \tilde{x} = \omega x/c_0, E = E_0\tilde{E}, T = T_0\tilde{T}, \tilde{\gamma}E_0/T_0 = \gamma, \tilde{\sigma}_0 = \sigma_0, \tilde{\sigma}_1 = T_0\sigma_1, \nu' = \nu\omega/c_0^2, \rho = \Lambda T_0^2/\omega$. Scaling Equations (7) and (8) and dropping the tildes , we obtain:

$$E_{tt} - \Delta E + \left(\frac{\sigma_0 + \sigma_1 T}{\omega}\right)E_t = 0, \tag{9}$$

$$T_t = \nu'\Delta T + \rho(T^2 - T + \gamma|E|)(1 - T). \tag{10}$$

To construct the steady hot-spot solution we assume that the monochromatic electric field $E = v\mathrm{e}^{\mathrm{i}qt}$ has formed a caustic along the plane curve $\mathbf{x}(s)$ as shown in Figure 2. Here q is a typical large (range of geometric optics) value of a nondimensional frequency. The first step is to introduce local coordinates s and λ, where λ is measured along the normal to the caustic and s is the arc length along it. We solve the equation for the electric field assuming high temperature around the caustic $\lambda = 0$. Once we have formed the electric field, the temperature field is determined, giving two-dimensional solutions of the heat equation with boundary layers along

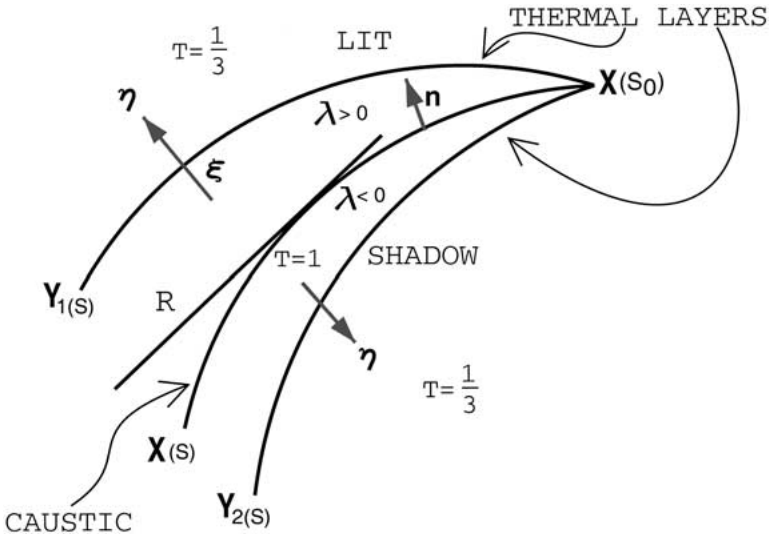

Figure 2. The caustic and the heat layers are indicated by the corresponding boundaries.

curves $\lambda = \lambda_1(s)$ and $\lambda = \lambda_2(s)$ which depend on the electric field and are to be found in the matching procedure. The temperature-layer solution finally is used to determine the electric field outside the high-temperature region.

We begin with the solution for the electric field. Taking $E = \mathrm{e}^{\mathrm{i}qt} v(\lambda, s)$ we obtain, in caustic coordinates, where $k(s)$ is the curvature of the caustic

$$\frac{1}{(1+\lambda k)}\left\{\frac{\partial}{\partial\lambda}\left[(1+\lambda k)\frac{\partial v}{\partial\lambda}\right]+\frac{\partial}{\partial s}\left[\frac{1}{(1+\lambda k)}\frac{\partial v}{\partial s}\right]\right\}+q^2 v - r\mathrm{i}qv = 0 \tag{11}$$

where $r = (\sigma_0 + \sigma_1 T)/\omega$, with $T = 1$ which is the highest hot-spot temperature. We now assume q to be relatively large with respect to the loss r and use the geometric-optics solution (see [7, section 1.2]) in the form:

$$\begin{aligned} v &= \mathrm{e}^{\mathrm{i}s(q^2-\mathrm{i}rq)^{1/2}} A(s, \tilde{\lambda}), \\ \tilde{\lambda} &= q^{2/3}\lambda, \end{aligned} \tag{12}$$

where the branch of the square root has a negative real part which is required by the damping of the field along the rays R of Figure 2, which are parametrized with positive increasing s. Substitution of (12) in (11) gives, to leading order in q, that the amplitude A satistifies the usual Airy equation

$$\frac{\partial^2}{\partial\tilde{\lambda}^2}A'' + 2\tilde{\lambda}kA = 0. \tag{13}$$

The region $\lambda > 0$ is lit, while the region $\lambda < 0$ is shadow. The solution for A is readily obtained in the form:

$$A = \ P\ A_i(-q^{2/3}\lambda\sqrt{|k(s)|}) \tag{14}$$

where A_i is the usual Airy function [7, p 52]. To determine the constant P we match (14) with the incoming field, which is taken to be $\hat{E}$. Also, assuming $r << 1$, we have the final expression for E in the neighbourhood of the caustic $\mathbf{x}(s)$ in the form:

$$E = \hat{E}\mathrm{e}^{+\mathrm{i}q(t-s)}\mathrm{e}^{-\frac{rs}{2q}}A_i(-q^{2/3}\lambda\sqrt{|k(s)|}). \tag{15}$$

We now determine the temperature field. To this end we recall from [4] and [8] that the one-dimensional nonlinear heat equation (10) admits steady solutions in the form of boundary layers. These solutions are explicitly obtained by integration of the corresponding ODE in the boundary-layer variable. These solutions are boundary layers decreasing from high-temperature regions to low temperature and *vice versa*, as can be readily seen from the exact solutions discussed in [4]. To construct the desired two-dimensional solutions we recall [8] that it is possible to construct, in the limit $\nu' \to 0$, two-dimensional boundary-layer solutions as follows. Assume the temperature boundary layer to be located along the curve $\mathbf{y}(\xi)$, where ξ is the arc length of such a curve. In local coordinates ξ and η with η normal to $\mathbf{y}\ (\xi)$ and κ the curvature of the temperature layer, the steady heat equation (10) takes the form:

$$\nu'\frac{1}{1+\eta\kappa}\left(\frac{\partial}{\partial\eta}(1+\kappa\eta)\frac{\partial T}{\partial\eta} + \frac{\partial}{\partial\xi}\frac{1}{1+\eta\kappa}\frac{\partial}{\partial\xi}T\right) + \rho(T^2 - T + \gamma|E|)(1-T) = 0. \tag{16}$$

The appropriate boundary-layer solution with transition at the curve $\mathbf{y}(\xi)$ can be either decreasing or increasing across the curve $\mathbf{y}(\xi)$ as a function of the normal variable η. Using a usual procedure [9, pp. 304–305] for the decreasing case, the solution is

$$T = \frac{1}{2}\left(\frac{1-(1-4\gamma|E|)^{1/2} + 2\mathrm{e}^{-\eta(\rho/2\nu')((1+(1-4\gamma E)^{1/2})/2)}}{1+\mathrm{e}^{-\eta(\rho/2\nu')((1+(1-4\gamma E)^{1/2})/2)}}\right), \tag{17}$$

where

$$\nu'\kappa(\xi) = \frac{1}{2}(\frac{1}{2}\nu'\rho)^{1/2}(1 - 3(1-4\gamma|E(\mathbf{y}(\xi))|^{1/2}). \tag{18}$$

Equation (18) is the condition for the curvature of the curve $\mathbf{y}(\xi)$ for the layer solution to be steady [8]. Similarly, the increasing solution has the same curvature equation, but now η is changed into $-\eta$.

To construct the hot-spot solution, we need to determine the curve $\mathbf{y}(\xi)$. From (18) to leading order $\mathbf{y}(\xi)$ is determined (since $\nu' << 1$) by the roots of the equation

$$1 = 3(1-4\gamma|E(\mathbf{y}(\xi))|^{1/2}).$$

This is the same equation as that obtained in one dimension in [4], but since now $E(\mathbf{y}(\xi))$ depends upon using the local coordinates ξ and η on Equation (18) provides a relationship between λ and s which defines the temperature boundary layer in terms of the caustic parameters. For this analysis to be valid we are assuming that the envelope of the electric field varies as in [4] on a slower scale compared to the width of the thermal layer. With this we have the implicit form of the temperature transition layers given by:

$$\frac{2}{9\gamma\hat{E}^2} = \mathrm{e}^{-\dfrac{qrs}{2}}A_i^2(-q^{2/3}\lambda\sqrt{|k(s)|}). \tag{19}$$

Since the maximum of A_i^2 is for $\lambda = 0$, Equation (19) has two solutions. We also have that the maximum penetration of the hot spot along the caustic is given by:

$$s_0 = -\frac{1}{qr}\log\left(\frac{2}{9\gamma\hat{E}^2}\right). \tag{20}$$

For $s \leq s_0$ Equation (20) defines two functions $\lambda_1(s)$ and $\lambda_2(s)$ which parametrize curves $\mathbf{y}_1(s)$ and $\mathbf{y}_2(s)$ whose graphs will join at $\mathbf{x}(s_0)$. These functions define the boundary of the hot spot. This geometry is shown in Figure 2.

Finally the profile for the temperature is given by

$$F(\xi, \eta) = \frac{1 + 3e^{-\eta\rho/2\nu'}}{3 + 3e^{-\rho_\eta/2\nu'}}, \tag{21}$$

where η is the coordinate normal to $\mathbf{y}_2(s)$. The same expression with η changed into $-\eta$ gives the temperature profile across $\mathbf{y}_1(s)$. The solution for the electric field can now be found outside the hot spot just by taking r in Equation (15) to depend on the temperature according to the solution (17) in the transition to the low-temperature region.

We have thus constructed a canonical two-dimensional hot-spot solution to Equations (7) and (8). The energy influx supplied by the incident field is balanced by the losses. The hot spot extent is determined by both the field envelope and the bistable nature of the temperature equation. The caustic region scales as the wave length which is larger as compared to the thermal boundary layer width.

It is to be noted that the hot spot is completely determined by the field at the initial caustic point, the damping rate along the caustic and the hotter and colder temperatures allowed by the material. The shape of the hot spot is controlled by the shape of the caustic, which is geometric and by the damping, which is a material property.

4. Conclusions

The experiments on microwave heating by millimetric waves show the formation of cusped two-dimensional hot-spot regions. The numerical experiments with ray-tracing codes coupled with the solutions for the heat equations reproduce the hot-spot patterns ([1], [2]). This evidence suggests a mechanism of hot-spot formation controlled by the caustics of the electromagnetic wave.

We have shown how the observations and numerical results reported in [1] and [2] can be explained in a simple way as the result of the geometry of the caustic, which in turn produces a cusped temperature-hot-spot pattern as the response to the high intensity at the caustic.

The results of this note provide simple formulae for calculating the extent and location of the hot spot in terms of the caustic geometry, which can be obtained by ray tracing. The results show that hot spots are unavoidable at caustics. Thus, to achieve uniform heating, the caustic must be elimitated. Finally these results are readily extended to three-dimensional situations and can provide a guide in the interpretation of numerical studies in more complex problems.

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